QM often uses complex functions

i is the square root of -1: $i^2 = -1$

observe: $i^0 = +1$, $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = +1$...

A complex number **z** has a real and imaginary part:

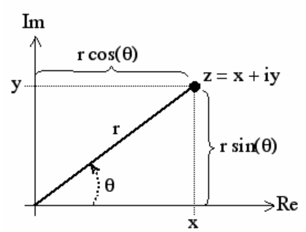
$\mathbf{z = x + iy}$

if y = 0, the number is real

if x = 0, it is purely imaginary

but in general it has both components

We can represent **z** as a point in the complex plane



QM often uses complex functions

The complex conjugate of z, denoted z*, has the opposite sign of y:

if $z = x + iy$, then $z^* = x - iy$

The absolute square of z, defined as z*z (and equal to zz*),

is real: $zz^* = (x+iy)(x-iy) = x^2 + ixy - ixy - i^2y = x^2+y^2$

This is the length of the vector r.

We can introduce polar coordinates:

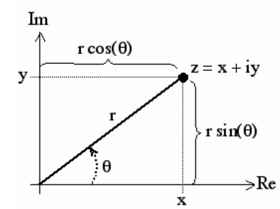
$x = r\cos\theta$, $y = r\sin\theta$

then $\mathbf{z = r(\cos\theta + i\sin\theta)}$

Trigonometic identity (see next slide):

$\mathbf{e^{i\theta} = (\cos\theta + i\sin\theta)}$ so $\mathbf{z = r\,e^{i\theta}}$

$z^*z = (r\,e^{-i\theta})(r\,e^{i\theta}) = r^2$ as before.



Trigonometric identities

$\cos x = 1 - x^2/2 + x^4/4! - \ldots = \sum (ix)^n/n!$ sum over even n.

note that the i^2 gives alternating signs.

$\sin x = x - x^3/3! + x^5/5! - \ldots = (1/i) \sum (ix)^n/n!$ sum over odd n.

$e^x = 1 + x + x^2/2 + x^3/3! = \sum x^n/n!$ sum over all n

Combine to get $\mathbf{e^{ix} = (\cos x + i \sin x)}$ **Euler's formula**

When $x = \pi$, this gives $\mathbf{e^{i\pi} = -1}$ or $\mathbf{e^{i\pi} + 1 = 0}$

Important related identities:

$\mathbf{\sin x = (e^{ix} - e^{-ix})/2i}$ and $\mathbf{\cos x = (e^{ix} + e^{-ix})/2}$

Just as we have complex numbers, we can have complex functions.

Observables (position, energy, momentum) must be real,

but the wavefunctions may be complex.

Complex functions may be easier to work with!

These identities on "Useful math facts" sheet.

Applying boundary conditions

$\psi_1(x) = A\sin(kx + \delta)$ two ways to write

$\psi_2(x) = C\,e^{+ikx} + D\,e^{-ikx}$ free-particle wavefunction

What do these look like for a particular set of boundary conditions?

Suppose we require that $\psi(x)=0$ at $x=0$:

$\psi_1(0) = A\sin(\delta) = 0$, so $\delta = 0$, so $\psi_1(x) = A\sin(kx)$.

$\psi_2(0) = C\,e^{+ikx} + D\,e^{-ikx} = C + D = 0$, thus $D = -C$

But that makes $\psi_2(x) = C\,[e^{+ikx} - e^{-ikx}] = 2iC\sin(kx)$

using the identity $\sin u = (e^{iu} - e^{-iu})/2i$

These are exactly the same function if $A = 2iC$, so $C = A/2i$.

Observe that this means that if A is real, C is imaginary.

$\psi_2^*(x)\psi_2(x) = (-A/2i)\sin(kx)\,(+A/2i)\sin(kx) = +A^2\sin^2(kx)$

which is greater than or equal to zero, as required.